Coordinates on Schubert cells, Kostant's harmonic forms, and the Bruhat-Poisson structure on G/B

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1 Introduction and Notation

This work grew out of my attempts to understand the relations between results of Kostant [K] on the de Rham cohomology of a flag manifold and Poisson geometry.

Let $X = G/B$ be a flag manifold, where G is a complex semi-simple Lie group and B is a Borel subgroup of G. Let K be a compact real form of G, so $X = G/B \cong K/T$, where $T = K \cap B$ is a maximal torus of K. In [K], Kostant constructs, for each element w in the Weyl group W, an explicit K-invariant closed differential form s^w on X with $\deg(s^w) = 2l(w)$, where $l(w)$ is the length of w (see Section 4). The cohomology classes of the s^w's form a basis of $H(X, \mathbb{C})$ that, up to constant multiples, is dual to the basis of the homology of X formed by the closures of the Bruhat (or Schubert) cells in X. These K-invariant forms on X are (d, ∂)-harmonic, where d is the de Rham differential operator and ∂ is a degree -1 operator introduced by Kostant.

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Our work was first motivated by wanting to understand the nature of the operator ∂ in terms of Poisson geometry.

The Poisson structure that is relevant to Kostant's theorem is the so-called Bruhat-Poisson structure on X [L-W]. It has its origin in the theory of quantum groups. It has the special property that its symplectic leaves are precisely the Bruhat cells in X (and hence the name). More properties of the Bruhat-Poisson structure are reviewed in Section 3.

The Bruhat-Poisson structure gives rise to the degree -1 operator

$$\partial_\pi \;=\; i_\pi d \;-\; d i_\pi$$

on the space of differential forms on X called the Koszul-Brylinski operator [Ko] [B], where π is the bi-vector field on X defining this Poisson structure, and i_π denotes the contraction operator of differential froms with π. It satisfies $\partial_\pi^2 = 0$. Its homology is called the Poisson homology of π. Poisson homology is closely related to cyclic homology of associative algebras, as is shown in [B]. It turns out that when restricted to K-invariant differential forms on X, Kostant's operator ∂ and the Koszul-Brylinski operator ∂_π differ by the contraction operator by a vector field θ_0 called the modular vector field of π (see [W2] [B-Z] [E-L-W]). More explicitly, it is the infinitesimal generator of the K-action on X in the direction of the element $2iH_\rho$ with ρ being half of the sum of all positive roots. This result, together with others on the Poisson (co)homology of the Bruhat-Poisson structure, can be found in [E-L].

In this paper, we relate Kostant's harmonic forms on X with the Bruhat-Poisson structure.

More precisely, the Bruhat-Poisson structure on $X = K/T$ is T-invariant (but not K-invariant). Thus, each Bruhat cell Σ_w inherits a T-invariant symplectic structure Ω_w. Use $\phi_w : \Sigma_w \to \mathfrak{t}^*$ to denote the moment map and let μ_w be the Liouville volume form on Σ_w defined by Ω_w. Theorem 4.3 says that when restricted to the cell Σ_w, Kostant's form s^w is related to the Liouville volume form μ_w by

$$s^w|_{\Sigma_w} \;=\; e^{\langle \phi_w,\; 2iH_\rho \rangle} \mu_w.$$

Notice that the function $\langle \phi_w,\; 2iH_\rho\rangle$ is the Hamiltonian function for the modular vector field θ_0 on Σ_w. (The vector field θ_0 is not globally Hamiltonian (see Section 4), but it is on each cell). Theorem 4.3 thus expresses Kostant's harmonic forms totally in terms of data coming from the Bruhat-Poisson structure. In particular, it shows that the integral $\int_{\Sigma_w} s^w$ is of the Duistermaat - Heckman type. Wanting to see this was another motivation for this work.

Theorem 4.3 is proved by writing everything down in some coordinates

$$\{z_1,\ \bar{z}_1,\ z_2,\ \bar{z}_2,\ \ldots\ , z_l,\ \bar{z}_l\}$$

on each Bruhat cell Σ_w, where $l = l(w)$. These coordinates are motivated by, but are independent of, the Bruhat-Poisson structure. Among the quantities that we write down explicitly in the coordinates are (see the later sections for the notation)

- (Theorem 3.4) the symplectic 2-form Ω_w on Σ_w and thus the Liouville volume form μ_w:

$$\begin{aligned}\Omega_w &= \sum_{j=1}^{l} \frac{i}{\ll \alpha_j,\ \alpha_j \gg} \frac{1}{1+|z_j|^2}\, dz_j \wedge d\bar{z}_j \\ \mu_w &= \prod_{j=1}^{l} \frac{i}{\ll \alpha_j,\ \alpha_j \gg} \frac{1}{1+|z_j|^2}\, dz_j \wedge d\bar{z}_j\end{aligned}$$

- (Theorem 3.4) the moment map for the T-action on $(\Sigma_w,\ \Omega_w)$:

$$\phi_w :\ \Sigma_w \longrightarrow \mathfrak{t}^* : \qquad \phi_w \ = \ \sum_{j=1}^{l} (-\frac{1}{2} \log(1+|z_j|^2) \check{H}_{\alpha_j})$$

- (Theorem 2.7) the Haar measure dn on the group $N_w = N \cap wN_-w^{-1}$ that parametrizes Σ_w:

$$dn \ = \ \left(\prod_{j=1}^{l} \frac{i \ll \rho,\ \beta_j \gg}{\pi \ll \beta_j,\ \beta_j \gg} \right) \prod_{j=1}^{l} (1+|z_j|^2)^{\frac{2 \ll \rho,\ \beta_j \gg}{\ll \beta_j,\ \beta_j \gg} - 1} dz_j \wedge d\bar{z}_j$$

- (Theorem 2.5) the A-component $a_w(n)$ in the Iwasawa decomposition of the element $\dot{w}^{-1} n \dot{w}$ for $n \in N_w$ (where $\dot{w}$ is any representative of w in K):

$$a_w(n) \ = \ \prod_{j=1}^{l} \exp(\frac{1}{2} \log(1+|z_j|^2) \check{H}_{\beta_j})$$

- (Theorem 4.3) Kostant's harmonic form s^w restricted to Σ_w:

$$s^w|_{\Sigma_w} \ = \ \prod_{j=1}^{l} \frac{i}{\ll \alpha_j,\ \alpha_j \gg} (1+|z_j|^2)^{-\frac{2 \ll \rho,\ \alpha_j \gg}{\ll \alpha_j,\ \alpha_j \gg} - 1} dz_j \wedge d\bar{z}_j.$$

By comparing these explicit formulas, we immediately get (Corollaries 3.5 and 3.6)

$$\begin{aligned}\phi_w &= Ad_{\dot{w}} \log a_w(n) \\ \mu_w &= \left(\prod_{j=1}^{l} \frac{\pi}{\ll \rho,\ \beta_j \gg} \right) a_w(n)^{-2\rho} dn.\end{aligned}$$

These relate the moment map ϕ_w and the Liouville volume form μ_w to the familiar map a_w and the Haar measure dn on N_w. This is desirable for understanding the geometry of the Bruhat-Poisson structure. The relation between the differential form $s^w|_{\Sigma_w}$ and the Liouville form μ_w also follows immediately from these formulas.

The coordinates $\{z_1,\ \bar{z}_1,\ z_2,\ \bar{z}_2,\ \ldots,z_l,\ \bar{z}_l\}$ are presented first in Section 2. Here we derive the formulas for the Haar measure dn of N_w and for the map $a_w : N \to A$. As one other application of these coordinates, we show that Harish-Chandra's formula for the c-function (in the case of a complex group) follows easily as a product of 1-dimensional integrals. Our calculation here is easier because we have pushed the usual induction argument on the length of w into the calculations for the explicit formulas for $a_w(n)$ and dn. The total amount of effort is probably the same.

In Section 3, we review some of the properties of the Bruhat-Poisson structure on K/T. We derive the formulas for the symplectic form Ω_w, thus also the Liouville measure μ_w, and the moment map for the T action on each Bruhat cell in the $\{z_1,\ \bar{z}_1,\ z_2,\ \bar{z}_2,\ \ldots,z_l,\ \bar{z}_l\}$-coordinates. By using the formulas for $a_w(n)$ and dn given in Section 2, we arrive at the (coordinate-free) interpretations for both the moment map ϕ_w and the Liouville measure μ_w in terms of $a_w(n)$ and dn as given in Corollaries 3.5 and 3.6.

Kostant's harmonic forms are reviewed in Section 4. Theorem 4.3 is given as an easy corollary of our formulas in the z-coordinates. Applications of Theorem 4.3 to the calculations of the Poisson (co)homology of the Bruhat-Poisson structure are given in [E-L].

In the Appendix, we discuss how our coordinates $\{z_1,\ \bar{z}_1,\ z_2,\ \bar{z}_2,\ \ldots,z_l,\ \bar{z}_l\}$ are related to the (complex) Bott-Samelson coordinates.

We now fix the notation.

Let G be a finite dimensional complex semi-simple Lie group with Lie algebra $\mathfrak{g}$. Let H be a Cartan subgroup and $\mathfrak{h}$ its Lie subalgebra. Denote by R the set of all roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let R^+ be a choice of positive roots. We will also write $\alpha > 0$ for $\alpha \in R^+$. Let $\mathfrak{b} = \mathfrak{b}_+$ be the Borel subalgebra spanned by $\mathfrak{h}$ and all the positive root vectors, and let B be the corresponding Borel subgroup.

Let W be the Weyl group of G relative to H. The Bruhat decomposition

$$G \;=\; \bigcup_{w \in W} B\dot{w}B,$$

where $\dot{w}$ is a representative of w in the normalizer of H in G, gives rise to the Bruhat decomposition

$$G/B \;=\; \bigcup_{w \in W} B\dot{w}B/B$$

of the flag manifold G/B into a disjoint union of cells. These cells, denoted by $\Sigma_w = B\dot{w}B/B$, are called Bruhat or Schubert cells, and their closures, $X_w = \overline{\Sigma_w}$, are called the Schubert varieties.

We choose a compact real form $\mathfrak{k}$ of $\mathfrak{g}$ as follows: let $\ll\gg$ be the Killing form of $\mathfrak{g}$. For each positive root α, denote by H_α the image of α under the isomorphism $\mathfrak{h}^* \to \mathfrak{h}$ via $\ll\,,\,\gg$, i.e., for any $H \in \mathfrak{h}$,

$$\ll H_\alpha,\ H \gg \;=\; \alpha(H).$$

Choose root vectors E_α and $E_{-\alpha}$ for α and $-\alpha$ respectively such that $\ll E_\alpha,\ E_{-\alpha} \gg \;=\; 1$. Then $[E_\alpha, E_{-\alpha}] = H_\alpha$. Set

$$X_\alpha \;=\; E_\alpha \;-\; E_{-\alpha}, \qquad Y_\alpha \;=\; i(E_\alpha \;+\; E_{-\alpha}). \tag{1}$$

The real subspace

$$\mathfrak{k} \;=\; \mathrm{span}_{\mathbb{R}}\{iH_\alpha,\ X_\alpha,\ Y_\alpha:\ \alpha > 0\}$$

is a compact real form of $\mathfrak{g}$. Let K be the corresponding compact subgroup of G. The intersection $T = K \cap B$ is a maximal torus of K, and its Lie algebra is

$$\mathfrak{t} \;=\; \mathrm{span}_{\mathbb{R}}\{iH_\alpha:\ \alpha > 0\}.$$

Let $\mathfrak{a} \;=\; i\mathfrak{t}$, and let $\mathfrak{n} = \mathfrak{n}_+$ be the subalgebra of $\mathfrak{g}$ spanned by all the positive root vectors. Let A and N be the corresponding subgroups. Then

$$G \;=\; KAN$$

is the Iwasawa decomposition of G as a real semi-simple Lie group. It follows that the projection map from $K \subset G$ to G/B induces an isomorphism from K/T to G/B. From now on, we will identify G/B with K/T this way.

For $g \in G$, let $g = k_g a_g n_g$ be the Iwasawa decomposition of g. The map

$$G \times K \longrightarrow K:\ (g,\ k) \longmapsto k_{gk} \tag{2}$$

defines a left action of G on K. We will denote this action by $(g, k) \to g \circ k$.

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2 The coordinates $\{z_1,\ \bar{z}_1,\ z_2,\ \bar{z}_2,\ \ldots,z_l,\ \bar{z}_l\}$ on N_w

Let $w \in W$ be a Weyl group element with length $l = l(w)$. Set

$$R_w^+ \ = \ \{\alpha > 0 : \ w^{-1}\alpha < 0\}$$

and

$$\mathfrak{n}_w \ = \ \mathrm{span}_{\mathbb{C}}\{E_\alpha : \ \alpha \in R_w^+\},$$

and let N_w be the subgroup of G with Lie algebra $\mathfrak{n}_w$. Then $N_w = N \cap wN_-w^{-1}$, where N_- is the "opposite" of N, and the map

$$j_w : \ N_w \longrightarrow \Sigma_w : \ n \longmapsto nw/B \tag{3}$$

is a holomorphic diffeomorphism. Thus, any complex coordinate system on N_w will give one on Σ_w. The coordinates $\{z_1,\ \bar{z}_1,\ z_2,\ \bar{z}_2,\ \ldots,z_l,\ \bar{z}_l\}$ that we will introduce in this section, however, will not be complex for a general w with $l(w) > 1$ (see Example 2.9). To obtain these coordinates, we make use of the compact real form K of G and the isomorphism of G/B and K/T. They are motivated by the Bruhat-Poisson structure on K/T.

Let $\dot{w}$ be a representative of w in K. For $n \in N_w$, let $a_w(n)$ be the A-component of the element $\dot{w}^{-1}n\dot{w}$ for the Iwasawa decomposition, i.e.,

$$\dot{w}^{-1}n\dot{w} \ = \ k_1 a_w(n) m_1 \tag{4}$$

for some $k_1 \in K$ and $m_1 \in N$. Notice that the map $a_w : N_w \to A$ depends only on w and not on the choice of $\dot{w}$.

In this section, we will write down the map $a_w : N_w \to A$ explicitly in the coordinates $\{z_1,\ \bar{z}_1,\ z_2,\ \bar{z}_2,\ \ldots,z_l,\ \bar{z}_l\}$ (Theorem 2.5). We will also write down the Haar measure of N_w in these coordinates (Theorem 2.7). As an application, we show how Harish-Chandra's formula for the c-function also follows easily as a product of 1-dimensional integrals.

We now describe the coordinates $\{z_1,\ \bar{z}_1,\ z_2,\ \bar{z}_2,\ \ldots,z_l,\ \bar{z}_l\}$.

Let again $\dot{w}$ be a representative of w in K. Let

$$C_{\dot{w}} \ = \ N_w \circ \dot{w}$$

be the N_w-orbit in K through $\dot{w}$ of the action $(n, k) \to n \circ k$. Recall that $n \circ k = k_1$ if $nk = k_1 b$ is the Iwasawa decomposition of nk with $k_1 \in K$ and $b \in AN$. The map

$$J_{\dot{w}} : \ N_w \longrightarrow C_{\dot{w}} : \ n \longmapsto n \circ \dot{w}$$

is then a diffeomorphism. Moreover, $J_{\dot{w}}$ followed by the projection map from $C_{\dot{w}} \subset K \subset G$ to G/B is just the map j_w in (3). Thus $C_{\dot{w}}$ is a lift of Σ_w in K.

The case of $l(w) = 1$. When $w = \sigma_\gamma$ is the simple reflection corresponding to a simple root γ, we use, for notational simplicity, N_γ to denote N_{σ_γ} and $\dot{\gamma}$ for $\dot{\sigma}_\gamma$. In this case, set

$$\check{H}_\gamma = \frac{2}{\ll \gamma, \gamma \gg} H_\gamma, \qquad \check{E}_\gamma = \sqrt{\frac{2}{\ll \gamma, \gamma \gg}}\, E_\gamma, \qquad \check{E}_{-\gamma} = \sqrt{\frac{2}{\ll \gamma, \gamma \gg}}\, E_{-\gamma}.$$

Then the map

$$\psi_\gamma : \ sl(2, \mathbb{C}) \longrightarrow \mathfrak{g} : \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longmapsto \check{H}_\gamma, \ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longmapsto \check{E}_\gamma, \ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \longmapsto \check{E}_{-\gamma} \tag{5}$$

is a Lie algebra homomorphism. It induces a Lie group homomorphism

$$\Psi_\gamma : \ SL(2, \mathbb{C}) \longrightarrow G,$$

and $\Psi_\gamma(SU(2)) \subset K$. For $z \in \mathbb{C}$, let

$$n_z = \Psi_\gamma \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = \exp(z \check{E}_\gamma) \in N_\gamma.$$

The map

$$\mathbb{C} \longrightarrow N_\gamma : \ z \longmapsto n_z$$

is clearly a parametrization of N_γ by $\mathbb{C}$. This is a complex coordinate system on N_γ.

Notice that the element $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ in $SU(2)$ is a representative of the non-trivial element in the Weyl group of $SL(2, \mathbb{C})$. Let

$$\dot{\gamma} = \Psi_\gamma \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in K.$$

It is a representative of σ_γ in K. The following Iwasawa decomposition in $SL(2, \mathbb{C})$

$$\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} \frac{iz}{\sqrt{1+|z|^2}} & \frac{i}{\sqrt{1+|z|^2}} \\ \frac{i}{\sqrt{1+|z|^2}} & \frac{-i\bar{z}}{\sqrt{1+|z|^2}} \end{pmatrix} \begin{pmatrix} \sqrt{1+|z|^2} & \frac{\bar{z}}{\sqrt{1+|z|^2}} \\ 0 & \frac{1}{\sqrt{1+|z|^2}} \end{pmatrix}$$

gives the Iwasawa decomposition of $n_z \dot{\gamma}$ in G as

$$n_z \dot{\gamma} = \Psi_\gamma \begin{pmatrix} \frac{iz}{\sqrt{1+|z|^2}} & \frac{i}{\sqrt{1+|z|^2}} \\ \frac{i}{\sqrt{1+|z|^2}} & \frac{-i\bar{z}}{\sqrt{1+|z|^2}} \end{pmatrix} \Psi_\gamma \begin{pmatrix} \sqrt{1+|z|^2} & \frac{\bar{z}}{\sqrt{1+|z|^2}} \\ 0 & \frac{1}{\sqrt{1+|z|^2}} \end{pmatrix}.$$

Thus

$$n_z \circ \dot{\gamma} \;=\; \Psi_\gamma \begin{pmatrix} \dfrac{iz}{\sqrt{1+|z|^2}} & \dfrac{i}{\sqrt{1+|z|^2}} \\ \dfrac{i}{\sqrt{1+|z|^2}} & \dfrac{-i\bar{z}}{\sqrt{1+|z|^2}} \end{pmatrix} \;\in\; K.$$

The map

$$\mathbb{C} \longrightarrow C_{\dot{\gamma}} : \; z \longmapsto n_z \circ \dot{\gamma} \;=\; \Psi_\gamma \begin{pmatrix} \dfrac{iz}{\sqrt{1+|z|^2}} & \dfrac{i}{\sqrt{1+|z|^2}} \\ \dfrac{i}{\sqrt{1+|z|^2}} & \dfrac{-i\bar{z}}{\sqrt{1+|z|^2}} \end{pmatrix}$$

is a parametrization of $C_{\dot{\gamma}}$ by $\{z, \; \bar{z}\}$. We also see that

$$a_{\sigma_\gamma}(n_z) \;=\; \exp(\frac{1}{2}\log(1+|z|^2)\check{H}_\gamma).$$

The general case. For a general element $w \in W$, let $l = l(w)$ be the length of w, and let

$$w \;=\; \sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_l} \tag{6}$$

be a reduced decomposition, where $\gamma_1, \gamma_2, ..., \gamma_l$ are simple roots. Again for notational simplicity, we use N_{γ_j} to denote $N_{\sigma_{\gamma_j}}$ for $j = 1, ..., l$. We now have the Lie group homomorphism

$$\Psi_{\gamma_j} : \; SL(2,\mathbb{C}) \longrightarrow G$$

for each j. Let again

$$\dot{\gamma}_j \;=\; \Psi_{\gamma_j} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \;\in\; K.$$

Write an element in N_{γ_j} as

$$n_{z_j} \;=\; \exp(z_j \check{E}_{\gamma_j})$$

for $z_j \in \mathbb{C}$. Set

$$\dot{w} \;=\; \dot{\gamma_1}\dot{\gamma_2}\cdots\dot{\gamma_l} \in K.$$

Theorem 2.1 *There is a diffeomorphism (between real manifolds)*

$$F_w : \; N_{\gamma_1} \times N_{\gamma_2} \times \cdots \times N_{\gamma_l} \longrightarrow N_w$$

characterized by

$$F_w(n_1, n_2, ..., n_l) \circ \dot{w} \;=\; (n_1 \circ \dot{\gamma_1})(n_2 \circ \dot{\gamma_2})\cdots(n_l \circ \dot{\gamma_l}) \;\in\; K. \tag{7}$$

The map

$$\mathbb{C}^l \longrightarrow N_w : \; (z_1, \; z_2, \; ..., \; z_l) \longmapsto F_w(n_{z_1}, \; n_{z_2}, \; ... \;, \; n_{z_l}) \tag{8}$$

gives coordinates $\{z_1, \; \bar{z}_1, \; z_2, \; \bar{z}_2, \; ... \;, z_l, \; \bar{z}_l\}$ *on* N_w *(as a real manifold).*

Remark 2.2 This is the same as saying that the map

$$C_{\dot{\gamma}_1} \times C_{\dot{\gamma}_2} \times \cdots \times C_{\dot{\gamma}_l} \longrightarrow C_{\dot{w}}$$

given by multiplication in K is a diffeomorphism. This statement is given in [S]. The proof we give below contains a recursive formula for F_w that will be used later.

Remark 2.3 We use $\{z_1,\ \bar{z}_1,\ z_2,\ \bar{z}_2,\ \ldots,z_l,\ \bar{z}_l\}$ instead of $\{z_1,\ z_2,\ \ldots,\ z_l\}$ to denote the coordinates to emphasize the fact that the map in (8) is in general not a holomorphic diffeomorphism. See Example 2.9.

Remark 2.4 Even though we use F_w to denote the map in Theorem 2.1, it depends not only on w but also on the reduced decomposition $w = \sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_l}$ for w. Therefore, the coordinates $\{z_1,\ \bar{z}_1,\ z_2,\ \bar{z}_2,\ \ldots,z_l,\ \bar{z}_l\}$ depend on the choice of the reduced decomposition.

Proof of Theorem 2.1. We need to show that for every point $(n_1, \ldots, n_l) \in N_{\gamma_1} \times \cdots \times N_{\gamma_l}$, there exists (a necessarily unique) $F(n_1, \ldots, n_l) \in N_w$ such that (7) is satisfied. We also need to show that each $n \in N_w$ arises this way. We prove this by induction on $l(w)$. When $l(w) = 1$, the map F_w is the identity map. Now for w with $l(w) > 1$ and with the reduced decomposition given in (6), set

$$w_1 \ = \ \sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_{l-1}}$$

so that $w = w_1\sigma_{\gamma_l}$. We wish to relate F_w and F_{w_1}. To this end, we first recall that $N_{w_1} \subset N_w$ [J]. In fact, the multiplication map in N gives a diffeomorphism

$$N_{w_1} \times \dot{w}_1 N_{\gamma_l} \dot{w}_1^{-1} \longrightarrow N_w,$$

where

$$\dot{w}_1 \ = \ \dot{\gamma}_1\dot{\gamma}_2\cdots\dot{\gamma}_{l-1}.$$

Now given $(n_1, n_2, \ldots, n_l) \in N_{\gamma_1} \times N_{\gamma_2} \times \cdots \times N_{\gamma_l}$, let

$$n' \ = \ F_{w_1}(n_1, n_2, \ldots, n_{l-1}) \in N_{w_1}$$

be such that

$$n' \circ (\dot{\gamma}_1\dot{\gamma}_2\cdots\dot{\gamma}_{l-1}) \ = \ (n_1 \circ \dot{\gamma}_1)(n_2 \circ \dot{\gamma}_2)\cdots(n_{l-1} \circ \dot{\gamma}_{l-1}) \ \in \ K.$$

We search for $x \in \dot{w}_1 N_{\gamma_l} \dot{w}_1^{-1}$ such that $n = n'x \in N_w$ satisfies

$$n \circ (\dot{\gamma}_1 \dot{\gamma}_2 \cdots \dot{\gamma}_l) \;=\; (n_1 \circ \dot{\gamma}_1)(n_2 \circ \dot{\gamma}_2) \cdots (n_l \circ \dot{\gamma}_l) \;\in\; K. \tag{9}$$

Now

$$\begin{aligned} n\dot{\gamma}_1 \cdots \dot{\gamma}_l &= n'x \; \dot{\gamma}_1 \cdots \; \dot{\gamma}_{l-1}\dot{\gamma}_l \\ &= (n'\dot{\gamma}_1 \cdots \dot{\gamma}_{l-1}) \; (\dot{w}_1^{-1} x \dot{w}_1) \; \dot{\gamma}_l \\ &= (n'\dot{\gamma}_1 \cdots \dot{\gamma}_{l-1}) \; x' \; \dot{\gamma}_l, \end{aligned}$$

where

$$x' \;=\; \dot{w}_1^{-1} x \dot{w}_1 \in N_{\gamma_l}.$$

In the notation we have introduced so far, we have

$$n'\dot{\gamma}_1 \cdots \dot{\gamma}_{l-1} \;=\; (n_1 \circ \dot{\gamma}_1)(n_2 \circ \dot{\gamma}_2) \cdots (n_{l-1} \circ \dot{\gamma}_{l-1}) a_{w_1}(n') m' \tag{10}$$

for some $m' \in N$. Thus

$$n\dot{\gamma}_1 \cdots \dot{\gamma}_l \;=\; (n_1 \circ \dot{\gamma}_1)(n_2 \circ \dot{\gamma}_2) \cdots (n_{l-1} \circ \dot{\gamma}_{l-1}) a_{w_1}(n') \; m' \; x' \; \dot{\gamma}_l.$$

Denote by $N_{\hat{\gamma}_l}$ the subgroup of N with Lie algebra

$$\mathfrak{n}_{\hat{\gamma}_l} \;=\; \mathrm{span}_{\mathbb{C}}\{E_\alpha : \; \alpha > 0, \; \alpha \neq \gamma_l\}.$$

Then the multiplication map in N induces a diffeomorphism $N_{\gamma_l} \times N_{\hat{\gamma}_l} \to N$, so we have the decomposition $N = N_{\gamma_l} N_{\hat{\gamma}_l}$. Moreover, ${\dot{\gamma}_l}^{-1} N_{\hat{\gamma}_l} \dot{\gamma}_l = N_{\hat{\gamma}_l}$. Thus, if we take x' to be the element $n^{\gamma_l} \in N_{\gamma_l}$ in the decomposition

$$(m')^{-1} a_{w_1}(n')^{-1} n_l a_{w_1}(n') \;=\; n^{\gamma_l} n^{\hat{\gamma}_l} \in N_{\gamma_l} N_{\hat{\gamma}_l}, \tag{11}$$

then

$$m'x' \;=\; a_{w_1}(n')^{-1} n_l a_{w_1}(n') (n^{\hat{\gamma}_l})^{-1},$$

and

$$\begin{aligned} n\dot{\gamma}_1 \cdots \dot{\gamma}_l &= (n_1 \circ \dot{\gamma}_1)(n_2 \circ \dot{\gamma}_2) \cdots (n_{l-1} \circ \dot{\gamma}_{l-1}) \; n_l \; a_{w_1}(n')(n^{\hat{\gamma}_l})^{-1} \; \dot{\gamma}_l \\ &= (n_1 \circ \dot{\gamma}_1)(n_2 \circ \dot{\gamma}_2) \cdots (n_{l-1} \circ \dot{\gamma}_{l-1}) n_l \dot{\gamma}_l \left({\dot{\gamma}_l}^{-1} a_{w_1}(n') \dot{\gamma}_l\right) \left({\dot{\gamma}_l}^{-1} (n^{\hat{\gamma}_l})^{-1} \dot{\gamma}_l\right). \end{aligned}$$

Set

$$m'' \;=\; {\dot{\gamma}_l}^{-1} (n^{\hat{\gamma}_l})^{-1} \dot{\gamma}_l \in N_{\hat{\gamma}}$$

and let

$$n_l\dot{\gamma}_l \;=\; (n_l \circ \dot{\gamma}_l)\; a_l m_l$$

be the Iwasawa decomposition for $n_l\dot{\gamma}_l$, so $a_l \in A$ and $m_l \in N_{\gamma_l}$. Then

$$n\dot{\gamma}_1 \cdots \dot{\gamma}_l \;=\; (n_1 \circ \dot{\gamma}_1)(n_2 \circ \dot{\gamma}_2) \cdots (n_l \circ \dot{\gamma}_l)\; a_l\; m_l \left(\dot{\gamma}_l{}^{-1} a_{w_1}(n')\dot{\gamma}_l\right) m'',$$

or

$$n\dot{\gamma}_1 \cdots \dot{\gamma}_l \;=\; (n_1 \circ \dot{\gamma}_1)(n_2 \circ \dot{\gamma}_2) \cdots (n_l \circ \dot{\gamma}_l)\; a_l\; (\dot{\gamma}_l{}^{-1} a_{w_1}(n')\dot{\gamma}_l)\; m''', \tag{12}$$

where

$$m''' \;=\; \left(\dot{\gamma}_l{}^{-1} a_{w_1}(n')\dot{\gamma}_l\right)^{-1} m_l \left(\dot{\gamma}_l{}^{-1} a_{w_1}(n')\dot{\gamma}_l\right) m'' \in N_{\gamma_l} N_{\hat{\gamma}_l} \;=\; N.$$

Therefore, with this choice of $x' \in N_{\gamma_l}$, the element $n = n'x = n'\dot{w}_1 x' \dot{w}_1^{-1} \in N_w$ satisfies (9). Notice that since N_{γ_l} normalizes $N_{\hat{\gamma}_l}$, we can first decompose $(m')^{-1} \in N$ with respect to the decomposition $N = N_{\gamma_l} N_{\hat{\gamma}_l}$ to get

$$(m')^{-1} \;=\; m^{\gamma_l} m^{\hat{\gamma}_l} \tag{13}$$

with $m^{\gamma_l} \in N_{\gamma}$ and $m^{\hat{\gamma}_l} \in N_{\hat{\gamma}_l}$. Then

$$x' \;=\; n^{\hat{\gamma}_l} \;=\; m^{\gamma_l} a_{w_1}(n')^{-1} n_l a_{w_1}(n') \in N_{\gamma_l}. \tag{14}$$

To summerize, we set

$$f:\; N_{w_1} \times N_{\gamma_l} \longrightarrow N_w:\; (n',\; n_l) \longmapsto n'\dot{w}_1(m^{\gamma_l} a_{w_1}(n')^{-1} n_l a_{w_1}(n'))\dot{w}_1^{-1} \;\in\; N_w, \tag{15}$$

and define

$$F_w(n_1, ..., n_l) \;=\; n \;=\; n'\dot{w}_1(m^{\gamma_l} a_{w_1}(n')^{-1} n_l a_{w_1}(n'))\dot{w}_1^{-1} \in N_w. \tag{16}$$

Then $F_w : N_{\gamma_1} \times N_{\gamma_2} \times \cdots \times N_{\gamma_l} \to N_w$ is a well-defined map and

$$F_w \;=\; f \circ (F_{w_1} \times id).$$

It is also clear now that F_w is a diffeomorphism.

Q.E.D.

Formula (12) also gives the following recursive formula for $a_w(n)$:

$$a_w(n) \;=\; a_l\; \dot{\gamma}_l{}^{-1} a_{w_1}(n')\dot{\gamma}_l, \tag{17}$$

where $n_l\dot{\gamma}_l = (n_l \circ \dot{\gamma}_l)a_l m_l$ is the Iwasawa decomposition for $n_l\dot{\gamma}_l$. Now for each $j = 1, ..., l$, let

$$n_j\dot{\gamma}_j \;=\; (n_j \circ \dot{\gamma}_j)\; a_j m_j$$

be the Iwasawa decomposition. We get from (17) that

$$a_w(n) \;=\; a_l(\dot{\gamma}_l^{-1} a_{l-1}\dot{\gamma}_l)(\dot{\gamma}_l^{-1}\dot{\gamma}_{l-1}^{-1} a_{l-2}\dot{\gamma}_{l-1}\dot{\gamma}_l)\cdots(\dot{\gamma}_l^{-1}\dot{\gamma}_{l-1}^{-1}\cdots\dot{\gamma}_2^{-1} a_1\dot{\gamma}_2\cdots\dot{\gamma}_{l-1}\dot{\gamma}_l), \tag{18}$$

or, since A is commutative,

$$\dot{w}a_w(n)\dot{w}^{-1} \;=\; \prod_{j=1}^{l} \sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_{j-1}}\sigma_{\gamma_j}(a_j).$$

We know that in the $\{z_1,\ \bar{z}_1,\ z_2,\ \bar{z}_2,\ \ldots\ , z_l,\ \bar{z}_l\}$ coordinates, a_j is given by

$$a_j \;=\; \exp(\frac{1}{2}(1+|z_j|^2)\check{H}_{\gamma_j})$$

for $j = 1, ..., l$. Thus

$$\begin{aligned}\sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_{j-1}}\sigma_{\gamma_j}(a_j) \;&=\; \sigma_{\gamma_2}\cdots\sigma_{\gamma_{j-1}}\exp(-\frac{1}{2}\log(1+|z_j|^2)\check{H}_{\gamma_j})\\ &=\; \exp(-\frac{1}{2}\log(1+|z_j|^2)\check{H}_{\alpha_j}),\end{aligned}$$

where, for each $j = 1, ..., l$,

$$\alpha_j \;=\; \sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_{j-1}}(\gamma_j). \tag{19}$$

Recall that

$$\{\alpha_1,\ \alpha_2,\ \cdots\ \alpha_l\} \;=\; R_w^+ \;=\; \{\alpha > 0:\ w^{-1}\alpha < 0\}.$$

Thus we have

$$\dot{w}a_w(n)\dot{w}^{-1} \;=\; \prod_{j=1}^{l}\exp(-\frac{1}{2}\log(1+|z_j|^2)\check{H}_{\alpha_j}). \tag{20}$$

Let

$$\beta_j \;=\; -w^{-1}\alpha_j \;=\; -\sigma_l\sigma_{l-1}\cdots\sigma_j(\gamma_j) \;=\; \sigma_l\sigma_{l-1}\cdots\sigma_{j+1}(\gamma_j), \tag{21}$$

i.e.,

$$\begin{aligned}\beta_1 \;&=\; \sigma_{\gamma_l}\sigma_{\gamma_{l-1}}\cdots\sigma_{\gamma_2}(\gamma_1)\\ \beta_2 \;&=\; \sigma_{\gamma_l}\sigma_{\gamma_{l-1}}\cdots\sigma_{\gamma_3}(\gamma_2)\\ &\cdots\\ \beta_{l-1} \;&=\; \sigma_{\gamma_l}(\gamma_{l-1})\\ \beta_l \;&=\; \dot{\gamma}_l.\end{aligned}$$

We then know that

$$\{\beta_1,\ \beta_2,\ \dots,\beta_l\} \ = \ R^+_{w^{-1}} \ = \ \{\beta > 0:\ w\beta < 0\}.$$

We also know that $\ll \beta_j,\ \beta_j \gg = \ll \alpha_j,\ \alpha_j \gg = \ll \gamma_j,\ \gamma_j \gg$ for each $j = 1, ..., l$. This fact will be used later.

The following theorem now follows immediately from (20).

Theorem 2.5 *In the $\{z_1,\ \bar{z}_1,\ z_2,\ \bar{z}_2,\ \dots,z_l,\ \bar{z}_l\}$-coordinates, the function a_w on N_w defined by (4) is explicitly given by*

$$a_w(n) \ = \ \prod_{j=1}^{l} \exp(\frac{1}{2}\log(1+|z_j|^2)\check{H}_{\beta_j}), \tag{22}$$

where the β_j's are given by (21).

Remark 2.6 This formula for $a_w(n)$ also follows from a product formula found by Doug Pickrell in [P].

We now look at the left invariant Haar measure on N_w.

Theorem 2.7 *In the $\{z_1,\ \bar{z}_1,\ z_2,\ \bar{z}_2,\ \dots,z_l,\ \bar{z}_l\}$-coordinates, a left invariant (and thus also bi-invariant) Haar measure on N_w is given as*

$$dn \ = \ \lambda_w \prod_{j=1}^{l} (1+|z_j|^2)^{\frac{2\ll \rho,\ \beta_j \gg}{\ll \beta_j,\ \beta_j \gg}-1} dz_j \wedge d\bar{z}_j, \tag{23}$$

where

$$\lambda_w \ = \ \frac{1}{(-2\pi i)^l}\prod_{j=1}^{l}\frac{2\ll \rho,\ \beta_j \gg}{\ll \beta_j,\ \beta_j \gg} \ = \ \prod_{j=1}^{l}\frac{i\ll \rho,\ \beta_j \gg}{\pi \ll \beta_j,\ \beta_j \gg} \tag{24}$$

is such that

$$\int_{N_w} a_w(n)^{-4\rho} dn \ = \ 1.$$

Proof. Again we prove by induction on $l(w)$. When $l(w) = 1$ so $w = \sigma_\gamma$ for a simple root γ, we have

$$dn_\gamma \ = \ -\frac{1}{2\pi i} dz_1 \wedge d\bar{z}_1.$$

Now for w with $l(w) = l > 1$, we use the same notation as that in the proof of Theorem 2.1. Let

$$N_{\alpha_l} \;=\; \dot{w}_1 N_{\gamma_l} \dot{w}_1^{-1}$$

be the subgroup of N with Lie algebra $\mathbb{C}E_{\alpha_l}$, where $\alpha_l = w_1(\gamma_l)$. Then the multiplication map

$$\mu : \; N_{w_1} \times N_{\alpha_l} \longrightarrow N : \; (n', \; n_{\alpha_l}) \longmapsto n' n_{\alpha_l}$$

is a diffeomorphism. Since N is unipotent, we have, under the map μ,

$$dn \;=\; \lambda \; dn' dn_{\alpha_l},$$

where λ is a constant to be determined later, and we take

$$dn_{\alpha_l} \;=\; du \wedge d\bar{u}$$

if N_{α_l} is parametrized by $\{u, \; \bar{u}\}$ via $n_{\alpha_l} = \exp(uE_{\alpha_l})$.

Consider now the parametrization of N_w by $\mathbb{C}^l$ via F_w. Write $n = F_w(z_1, \bar{z}_1, ..., z_l, \bar{z}_l)$ if $n = F_w(n_1, ..., n_l)$ where $n_j = \exp(z_j \check{E}_{\gamma_j})$ for each j. Let again

$$n' \;=\; F_{w_1}(z_1, \; \bar{z}_1, \; \ldots \;, z_{l-1}, \; \bar{z}_{l-1}) \in N_{w_1}.$$

Recall that the element $m' \in N$ is given in (10) and that $m^{\gamma_l} \in N_{\gamma_l}$ is given in (13). Write

$$m^{\gamma_l} \;=\; \exp(m(z_1, \bar{z}_1, ..., z_{l-1}, \bar{z}_{l-1})\check{E}_{\gamma_l}).$$

Then we know from (16) that

$$n \;=\; n' \exp\left((m(z_1, \bar{z}_1, ..., z_{l-1}, \bar{z}_{l-1}) + a_{w_1}(n')^{-\gamma_l} z_l) v_l E_{\alpha_l}\right),$$

where $v_l \in \mathbb{C}$ is such that $Ad_{\dot{w}_1} \check{E}_{\gamma_l} = v_l E_{\alpha_l}$. Set

$$u \;=\; v_l(m(z_1, \bar{z}_1, ..., z_{l-1}, \bar{z}_{l-1}) + a_{w_1}(n')^{-\gamma_l} z_l).$$

Assume that dn' is given as in the theorem for w_1. This means, noting the definition of the β_j's, that

$$\begin{aligned} dn' \;&=\; \lambda_{w_1} \prod_{j=1}^{l-1} (1 + |z_j|^2)^{\frac{2 \ll \rho, \; \sigma_{\gamma_l}\beta_j \gg}{\ll \sigma_{\gamma_l}\beta_j, \; \sigma_{\gamma_l}\beta_j \gg} - 1} dz_j \wedge d\bar{z}_j \\ &=\; \lambda_{w_1} \prod_{j=1}^{l-1} (1 + |z_j|^2)^{-\frac{2 \ll \gamma_l, \; \beta_j \gg}{\ll \beta_j, \; \beta_j \gg}} (1 + |z_j|^2)^{\frac{2 \ll \rho, \; \beta_j \gg}{\ll \beta_j, \; \beta_j \gg} - 1} dz_j \wedge d\bar{z}_j. \end{aligned}$$

Here we have just used the fact that $\rho - \sigma_{\gamma_l}\rho = \gamma_l$. On the other hand, by Theorem 2.5, we have

$$a_{w_1}(n') = \prod_{j=1}^{l-1} \exp(\frac{1}{2}\log(1+|z_j|^2)\check{H}_{\sigma_{\gamma_l}(\beta_j)}),$$

so

$$a_{w_1}(n')^{-2\gamma_l} = \prod_{j=1}^{l-1} (1+|z_j|^2)^{\frac{\ll -2\gamma_l,\ \sigma_{\gamma_l}\beta_j \gg}{\ll \sigma_{\gamma_l}\beta_j,\ \sigma_{\gamma_l}\beta_j \gg}} = \prod_{j=1}^{l-1} (1+|z_j|^2)^{\frac{2\ll \gamma_l,\ \beta_j \gg}{\ll \beta_j,\ \beta_j \gg}}.$$

Therefore,

$$\begin{aligned} dn &= \lambda\ dn'\ dn_{\alpha_l} = \lambda'\ a_{w_1}(n')^{-2\gamma_l}\ dn'\ (dz_l \wedge d\bar{z}_l) \\ &= \lambda'\lambda_{w_1} \left(\prod_{j=1}^{l-1} (1+|z_j|^2)^{\frac{2\ll \rho,\ \beta_j \gg}{\ll \beta_j,\ \beta_j \gg} - 1} dz_j \wedge d\bar{z}_j \right) \wedge (dz_l \wedge d\bar{z}_l), \end{aligned}$$

where $\lambda' = \lambda|v_l|^2$ is a new constant to be determined later. Since $\beta_l = \gamma_l$ is a simple root, we have

$$\frac{2\ll \rho,\ \gamma_l \gg}{\ll \gamma_l,\ \gamma_l \gg} = 1.$$

Thus

$$dn = \lambda'\lambda_{w_1} \prod_{j=1}^{l} (1+|z_j|^2)^{\frac{2\ll \rho,\ \beta_j \gg}{\ll \beta_j,\ \beta_j \gg} - 1} dz_j \wedge d\bar{z}_j.$$

Using Theorem 2.5, we see that the integral

$$\int_N a_w(n)^{-4\rho} dn$$

is now a product of 1-dimensional ones and is easily calculated. The constant $\lambda_w = \lambda'\lambda_{w_1}$ must be given by (24) for the above integral to be equal to 1.

Q.E.D.

Example 2.8 We recall that for the complex group G considered as a real Lie group, the c-**function** for a Weyl group element w is defined to be (see [H], Chapter IV, §6)

$$c_w(\lambda) = \int_{\bar{N}_w} a(\bar{n})^{-(i\lambda+2\rho)} d\bar{n},$$

where $\bar{N}_w \subset N_-$ is the "opposite" of N_w and $a(\bar{n})$ is the A-component in the Iwasawa decomposition of $\bar{n} \in \bar{N}_w$. In our notation, we have

$$c_{w^{-1}}(\lambda) = \int_{N_w} a_w(n)^{-(i\lambda+2\rho)} dn.$$

Using our formulas for $a_w(n)$ and for dn in the $\{z_1,\ \bar{z}_1,\ z_2,\ \bar{z}_2,\ \dots,z_l,\ \bar{z}_l\}$ coordinates, one immediately reduces the integral to a product of 1-dimensional ones and gets

$$\begin{aligned}
c_{w^{-1}}(\lambda) &= \lambda_w \prod_{j=1}^{l(w)} \int_{\mathbb{R}^2} (1+|z_j|^2)^{-\dfrac{\ll i\lambda+2\rho,\ \beta_j \gg}{\ll \beta_j,\ \beta_j \gg}+\dfrac{2\ll \rho,\ \beta_j \gg}{\ll \beta_j,\ \beta_j \gg}-1} dz_j \wedge d\bar{z}_j \\
&= \lambda_w \prod_{j=1}^{l(w)} \int_{\mathbb{R}^2} (1+x_j^2+y_j^2)^{-\dfrac{\ll i\lambda,\ \beta_j \gg}{\ll \beta_j,\ \beta_j \gg}-1} (-2i)dx_j dy_j \\
&= \lambda_w \prod_{j=1}^{l(w)} (\ -2\pi i) \int_0^\infty (1+r_j^2)^{-\dfrac{\ll i\lambda,\ \beta_j \gg}{\ll \beta_j,\ \beta_j \gg}-1} 2r_j dr_j \\
&= (-2\pi i)^{l(w)}\ \lambda_w \prod_{j=1}^{l(w)} \frac{\ll \beta_j,\ \beta_j \gg}{\ll i\lambda,\ \beta_j \gg} \qquad \text{if } \mathrm{Re} \ll i\lambda,\ \beta_j \gg\ > 0 \ \text{ for each } j \\
&= \prod_{j=1}^{l(w)} \frac{\ll 2\rho,\ \beta_j \gg}{\ll i\lambda,\ \beta_j \gg} \qquad \text{if } \mathrm{Re} \ll i\lambda,\ \beta_j \gg\ > 0 \ \text{ for each } j.
\end{aligned}$$

This is the well-known formula of Harish-Chandra (see Theorem 5.7 in [H]). As we have mentioned in the Introduction, our calculation here is easier because we have pushed the induction argument that is normally used in the calculations for the c-functions into the calculations for $a_w(n)$ and dn.

Example 2.9 Consider the example of $\mathfrak{g} = sl(3,\mathbb{C})$. We take w be to the longest Weyl group element $w_0 = (1,2)(2,3)(1,2)$. In this case, parametrize $N_{w_0} = N$ by complex coordinates

$$(u_1,\ u_2,\ u_3) \longmapsto n \ = \ \begin{pmatrix} 1 & u_1 & u_3 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have

$$\dot{\gamma}_1 \ = \ \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \dot{\gamma}_2 \ = \ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \qquad \dot{\gamma}_3 \ = \ \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so

$$\dot{w} \ = \ \dot{\gamma}_1\dot{\gamma}_2\dot{\gamma}_3 \ = \ \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

The Iwasawa decomposition of $n\dot{w}$ in $SL(3,\mathbb{C})$ is

$$n\dot{w} = \begin{pmatrix} -u_3 & -u_1 & -1 \\ -u_2 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} -\dfrac{u_3}{\Delta_3} & -\dfrac{u_1(1+|u_2|^2)-\bar{u}_2 u_3}{\Delta_2\Delta_3} & -\dfrac{1}{\Delta_2} \\ -\dfrac{u_2}{\Delta_3} & -\dfrac{1+|u_2|^2-u_1u_2\bar{u}_3}{\Delta_2\Delta_3} & \dfrac{\bar{u}_1}{\Delta_2} \\ -\dfrac{1}{\Delta_3} & \dfrac{\bar{u}_2+u_1\bar{u}_3}{\Delta_2\Delta_3} & \dfrac{\bar{u}_3-\bar{u}_1\bar{u}_2}{\Delta_2} \end{pmatrix} \begin{pmatrix} \Delta_3 & \dfrac{\bar{u}_2+u_1\bar{u}_3}{\Delta_3} & \dfrac{\bar{u}_3}{\Delta_3} \\ 0 & \dfrac{\Delta_2}{\Delta_3} & \dfrac{\bar{u}_1(1+|u_2|^2)-u_2\bar{u}_3}{\Delta_2\Delta_3} \\ 0 & 0 & \dfrac{1}{\Delta_2} \end{pmatrix}$$

where

$$\Delta_1 = \sqrt{1+|u_1|^2}, \qquad \Delta_2 = \sqrt{1+|u_1|^2+|u_1u_2-u_3|^2}, \qquad \Delta_3 = \sqrt{1+|u_2|^2+|u_3|^2}.$$

On the other hand, for z_1, z_2 and z_3 in $\mathbb{C}$, let

$$n_1 = \begin{pmatrix} 1 & z_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad n_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \qquad n_3 = \begin{pmatrix} 1 & z_3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have

$$\begin{aligned} &(n_1 \circ \dot{\gamma}_1)(n_2 \circ \dot{\gamma}_2)(n_3 \circ \dot{\gamma}_3) \\ &= \frac{1}{\varepsilon_1\varepsilon_2\varepsilon_3} \begin{pmatrix} iz_1 & i & 0 \\ i & -i\bar{z}_1 & 0 \\ 0 & 0 & \varepsilon_1 \end{pmatrix} \begin{pmatrix} \varepsilon_2 & 0 & 0 \\ 0 & iz_2 & i \\ 0 & i & -i\bar{z}_2 \end{pmatrix} \begin{pmatrix} iz_3 & i & 0 \\ i & -i\bar{z}_3 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} \\ &= \frac{1}{\varepsilon_1\varepsilon_2\varepsilon_3} \begin{pmatrix} -\varepsilon_2 z_1 z_3 - iz_2 & -\varepsilon_2 z_1 + iz_2\bar{z}_3 & -\varepsilon_3 \\ -\varepsilon_2 z_3 + i\bar{z}_1 z_2 & -\varepsilon_2 - i\bar{z}_1 z_2 \bar{z}_3 & \bar{z}_1\varepsilon_3 \\ -\varepsilon_1 & \varepsilon_1\bar{z}_3 & -i\varepsilon_1\varepsilon_3\bar{z}_2 \end{pmatrix}, \end{aligned}$$

where $\varepsilon_j = \sqrt{1+|z_j|^2}$ for $j = 1,2,3$. By setting

$$n \circ \dot{w} = (n_1 \circ \dot{\gamma}_1)(n_2 \circ \dot{\gamma}_2)(n_3 \circ \dot{\gamma}_3),$$

we get the following coordinate change between our coordinates $\{(z_1, \bar{z}_1, z_2, \bar{z}_2, z_3, \bar{z}_3\}$ and the u's:

$$u_1 = z_1, \qquad u_2 = \frac{\varepsilon_2 z_3 - i\bar{z}_1 z_2}{\varepsilon_1}, \qquad u_3 = \frac{\varepsilon_2 z_1 z_3 + iz_2}{\varepsilon_1},$$

or

$$z_1 = u_1, \qquad z_2 = i\frac{u_1u_2 - u_3}{\Delta_1}, \qquad z_3 = \frac{\bar{u}_1 u_3 + u_2}{\Delta_2}.$$

Notice that this is not a holomorphic change of coordinates. Thus the $\{z_1, \bar{z}_1, z_2, \bar{z}_2, z_3, \bar{z}_3\}$ coordinates are not complex. Now in the u-coordinates, we have

$$a_w(n) \;=\; \begin{pmatrix} \Delta_3 & 0 & 0 \\ 0 & \dfrac{\Delta_2}{\Delta_3} & 0 \\ 0 & 0 & \dfrac{1}{\Delta_2} \end{pmatrix}.$$

Under the coordinate change, we have

$$\Delta_1 \;=\; \varepsilon_1, \qquad \Delta_2 \;=\; \varepsilon_1\varepsilon_2, \qquad \Delta_3 \;=\; \varepsilon_2\varepsilon_3.$$

Thus we get $a_w(n)$ in the $\{(z_1, \bar{z}_1, z_2, \bar{z}_2, z_3, \bar{z}_3\}$ coordinates as:

$$a_w(n) \;=\; \begin{pmatrix} \varepsilon_2\varepsilon_3 & 0 & 0 \\ 0 & \dfrac{\varepsilon_1}{\varepsilon_3} & 0 \\ 0 & 0 & \dfrac{1}{\varepsilon_1\varepsilon_2} \end{pmatrix}.$$

This is the same as what one would obtain from Theorem 2.5. Similarly, the left invariant Haar measure dn is, up to a constant multiple, given in the u-coordinates by

$$dn \;=\; du_1 \wedge d\bar{u}_1 \wedge du_2 \wedge d\bar{u}_2 \wedge du_3 \wedge d\bar{u}_3.$$

After the change of coordinates, we get

$$dn \;=\; \varepsilon_2^2 dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge dz_3 \wedge d\bar{z}_3.$$

Again, this is the same as what one would obtain from Theorem 2.7.

The following proposition will be used in Section 4.

Proposition 2.10 *We have*

$$F_w(e, ..., e, n_j, e, ..., e) \;=\; (\dot{\gamma}_1\dot{\gamma}_2 \cdots \dot{\gamma}_{j-1})\; n_j\; (\dot{\gamma}_1\dot{\gamma}_2 \cdots \dot{\gamma}_{j-1})^{-1} \in N_{\alpha_j}$$

for $j = 1, ..., l$ *and* $n_j \in N_j$, *and*

$$(F_w)_*(0)\left(\frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial \bar{z}_1} \wedge \cdots \wedge \frac{\partial}{\partial z_l} \wedge \frac{\partial}{\partial \bar{z}_l}\right) \;=\; (\prod_{j=1}^{l} \frac{i}{\ll \alpha_j,\; \alpha_j \gg}) E_{\alpha_1} \wedge iE_{\alpha_1} \wedge \cdots \wedge E_{\alpha_l} \wedge iE_{\alpha_l},$$

where $(F_w)_*(0)$ *is the differential of* F_w *at* 0.

Proof. Set $n = (\dot\gamma_1\dot\gamma_2\cdots\dot\gamma_{j-1})n_j(\dot\gamma_1\dot\gamma_2\cdots\dot\gamma_{j-1})^{-1}$. Let

$$n_j\dot\gamma_j \;=\; (n_j\circ\dot\gamma_j)\; a_j\; m_j$$

be the Iwasawa decomposition of $n_j\dot\gamma_j$, where $m_j\in N_{\gamma_j}$. Then

$$n_j\dot\gamma_j\dot\gamma_{j+1}\cdots\dot\gamma_l \;=\; (n_j\circ\dot\gamma_j)(\dot\gamma_{j+1}\cdots\dot\gamma_l)(\dot\gamma_{j+1}\cdots\dot\gamma_l)^{-1}a_j\;(\dot\gamma_{j+1}\cdots\dot\gamma_l)(\dot\gamma_{j+1}\cdots\dot\gamma_l)^{-1}m_j(\dot\gamma_{j+1}\cdots\dot\gamma_l)$$

Set

$$\begin{aligned} a'_j &= (\dot\gamma_{j+1}\cdots\dot\gamma_l)^{-1}a_j(\dot\gamma_{j+1}\cdots\dot\gamma_l)\in A \\ m'_j &= (\dot\gamma_{j+1}\cdots\dot\gamma_l)^{-1}m_j(\dot\gamma_{j+1}\cdots\dot\gamma_l)\in N. \end{aligned}$$

Then

$$n\;\dot\gamma_1\cdots\dot\gamma_l \;=\; \dot\gamma_1\cdots\dot\gamma_{j-1}(n_j\circ\dot\gamma_j)\;(\dot\gamma_{j+1}\cdots\dot\gamma_l)\;a'_j\;m'_j.$$

Thus

$$n\circ(\dot\gamma_1\cdots\dot\gamma_l) \;=\; \dot\gamma_1\cdots\dot\gamma_{j-1}(n_j\circ\dot\gamma_j)\;\dot\gamma_{j+1}\cdots\dot\gamma_l$$

Hence

$$F_w(e,...,e,n_j,e,...,e) \;=\; n.$$

Write $n_j \;=\; \exp(z_j\check{E}_{\gamma_j})$. Then

$$\begin{aligned} (\dot\gamma_1\dot\gamma_2\cdots\dot\gamma_{j-1})\;n_j\;(\dot\gamma_1\dot\gamma_2\cdots\dot\gamma_{j-1})^{-1} &= \exp(z_jAd_{\dot\gamma_1\dot\gamma_2\cdots\dot\gamma_{j-1}}(\check{E}_{\gamma_j})) \\ &= \exp(\sqrt{\frac{2}{\ll\gamma_j,\;\gamma_j\gg}}z_jAd_{\dot\gamma_1\dot\gamma_2\cdots\dot\gamma_{j-1}}(E_{\gamma_j})). \end{aligned}$$

Since $\sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_{j-1}}(\gamma_j)=\alpha_j$ by definition, we have

$$Ad_{\dot\gamma_1\dot\gamma_2\cdots\dot\gamma_{j-1}}(E_{\gamma_j}) \;=\; c_jE_{\alpha_j}$$

for some complex number c_j. Write $z_j = x_j+iy_j$ and $c_j = u_j+iv_j$. Using the fact that $\ll\gamma_j,\;\gamma_j\gg=\ll\alpha_j,\;\alpha_j\gg$, we get

$$\begin{aligned} (F_w)_*(0)(\frac{\partial}{\partial z_j}\wedge\frac{\partial}{\partial\bar z_j}) &= \frac{i}{2}(F_w)_*(0)(\frac{\partial}{\partial x_j}\wedge\frac{\partial}{\partial y_j}) \\ &= \frac{i}{\ll\alpha_j,\;\alpha_j\gg}(u_jE_{\alpha_j}+v_j(iE_{\alpha_j}))\wedge(-v_jE_{\alpha_j}+u_j(iE_{\alpha_j})) \\ &= \frac{i}{\ll\alpha_j,\;\alpha_j\gg}|c_j|^2\;E_{\alpha_j}\wedge iE_{\alpha_j}. \end{aligned}$$

Since both root vectors E_{γ_j} and E_{α_j} have length 1 with respect to the K-invariant Hermitian form on $\mathfrak{g}$ induced by $\mathfrak{k}$, we have $|c_j|^2=1$. The statement about $(F_w)_*(0)$ now follows immediately from this.

Q.E.D.

We now look at the T-action on N_w by conjugations:

$$T \times N_w \longrightarrow N_w : \ (t,\ n) \longmapsto t\ n\ t^{-1}.$$

For a given $t \in T$, set

$$\begin{aligned} t_1 &= t \\ t_2 &= \dot{\gamma}_1\ t\ \dot{\gamma}_1^{-1} \\ t_3 &= \dot{\gamma}_2\dot{\gamma}_1\ t\ (\dot{\gamma}_2\dot{\gamma}_1)^{-1} \\ & \cdots \\ t_l &= (\dot{\gamma}_{l-1}\dot{\gamma}_{l-2}\cdots\dot{\gamma}_1)\ t\ (\dot{\gamma}_{l-1}\dot{\gamma}_{l-2}\cdots\dot{\gamma}_1)^{-1}. \end{aligned}$$

Equip $N_{\gamma_1} \times N_{\gamma_2} \times \cdots \times N_{\gamma_l}$ with the T-action given by

$$t \cdot (n_1,\ n_2,\ ...,\ n_l) \ = \ (t_1 n_1 t_1^{-1},\ t_2 n_2 t_2^{-1},\ \cdots,\ t_l n_l t_l^{-1}). \tag{25}$$

Proposition 2.11 *1) With respect to the T-actions on N_w by conjugation and on $N_{\gamma_1} \times N_{\gamma_2} \times \cdots \times N_{\gamma_l}$ as given by (25), the map F_w is T-equivariant;*

2) In the $\{z_1,\ \bar{z}_1,\ z_2,\ \bar{z}_2,\ ...\ , z_l,\ \bar{z}_l\}$ coordinates, the T-action on N_w is given by

$$t \cdot (z_1,\ \bar{z}_1,\ ...\ ,\ z_l,\ \bar{z}_l) \ = \ (t^{\alpha_1} z_1,\ t^{-\alpha_1}\bar{z}_1,\ ...\ ,\ t^{\alpha_l} z_l,\ t^{-\alpha_l}\bar{z}_l), \tag{26}$$

where, recall, $\alpha_j = \sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_{j-1}}(\gamma_j)$ for $1 \leq j \leq l$.

Proof. For each $j = 1, 2, ..., l$, we have

$$(t_j n_j t_j^{-1}) \circ \dot{\gamma}_j \ = \ t_j (n_j \circ \dot{\gamma}_j)(\dot{\gamma}_j\ t_j^{-1}\ \dot{\gamma}_j^{-1}) \ = \ t_j (n_j \circ \dot{\gamma}_j) t_{j+1}^{-1},$$

where t_{l+1} is defined to be equal to $(\dot{\gamma}_l\dot{\gamma}_{l-1}\cdots\dot{\gamma}_1)\ t\ (\dot{\gamma}_l\dot{\gamma}_{l-1}\cdots\dot{\gamma}_1)^{-1} = \dot{w}^{-1} t \dot{w}$. Thus,

$$\prod_{j=1}^{l} (t_j n_j t_j^{-1}) \circ \dot{\gamma}_j \ = \ t \left(\prod_{j=1}^{l} (n_j \circ \dot{\gamma}_j) \right) \dot{w}^{-1} t \dot{w}.$$

On the other hand, let $n = F_w(n_1, ..., n_l) \in N_w$. We have

$$(t\ n\ t^{-1}) \circ \dot{w} \ = \ t(n \circ \dot{w})(\dot{w}^{-1} t \dot{w}).$$

It now follows from the definition of F_w that

$$F_w(t_1 n_1 t_1^{-1}, \ \cdots \ t_l n_l t_l^{-1}) \ = \ t \ n \ t^{-1}.$$

Write $t = \exp(iH) \in T$ for $H \in \mathfrak{a}$ and $n_j = \exp(z_j \check{E}_{\gamma_j}) \in N_{\gamma_j}$. Then (26) follows from the following calculation:

$$\begin{aligned} t_j \ n_j \ t_j^{-1} &= \exp(z_j \ Ad_{t_j} \check{E}_{\gamma_j}) \\ &= \exp(z_j \ e^{i\sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_{j-1}}(\gamma_j)(H)} \check{E}_{\gamma_j}) \\ &= \exp(t^{\alpha_j} z_j \ \check{E}_{\gamma_j}). \end{aligned}$$

Q.E.D.

3 The Bruhat-Poisson structure

Recall that a Poisson structure on a manifold M ([W1]) is a bivector field π on M such that the bracket operation on the algebra $C^\infty(M)$ of smooth functions on M defined by

$$\{\phi, \varphi\} \ = \ \pi(d\phi, d\varphi), \qquad \phi, \varphi \in C^\infty(M)$$

satisfies Jacobi's identity. The condition on π is that

$$[\pi, \ \pi] \ = \ 0,$$

where [,] denotes the Schouten bracket on the space of multivector fields on M ([Ko]).

The bivector field π can also be regarded as the bundle map

$$\tilde{\pi} : T^*M \longrightarrow TM : \ (\tilde{\pi}(\alpha), \ \beta) \ = \ \pi(\alpha, \ \beta).$$

When π is of maximal rank (M is then necessarily even dimensional), the bundle map $\tilde{\pi}$ is invertible, and the 2-form ω on M defined by $\omega(x, y) = \pi(\tilde{\pi}^{-1}(x), \ \tilde{\pi}^{-1}(y))$ is closed and non-degenerate and is thus a symplectic 2-form. In general, the image of π defines a (generally singular) involutive distribution on M. It has integrable submanifolds which inherit symplectic structures [W1]. They are called the symplectic leaves of π in P. Therefore, symplectic manifolds are special cases of Poisson manifolds and every Poisson manifold is a disjoint union of symplectic manifolds.

The Bruhat-Poisson structure on K/T comes from a Poisson structure π on K defined by

$$\pi \ = \ \Lambda^r \ - \ \Lambda^l,$$

where

$$\Lambda \ = \ \frac{1}{2}\sum_{\alpha>0} X_\alpha \wedge Y_\alpha \ \in \ \mathfrak{k}\wedge\mathfrak{k}$$

with X_α and Y_α given by (1), and Λ^r (resp. Λ^l) is the right (resp. left) invariant bi-vector field on K with value Λ at the identity element e. We summerize some properties of π in the next theorem. For details, see [STS] [S] and [L-W].

Theorem 3.1 *a) The bi-vector field π defines a Poisson structure on K;*

b) Equip $K \times K$ with the product Poisson structure $\pi \oplus \pi$. Then the multiplication map

$$K \times K \longrightarrow K: \ (k_1, \ k_2) \longmapsto k_1 k_2$$

is a Poisson map, making (K, π) into a Poisson Lie group;

c) The symplectic leaves of π in K are precisely the orbits of AN in K for the action given by (2). These are the Bruhat cells in K: for each Weyl group element $w \in W$ with a fixed representative $\dot{w}$ of w in K, the symplectic leaf through $\dot{w}$ is the N_w-orbit $C_{\dot{w}} = N_w \circ \dot{w}$ introduced in Section 2. For each $t \in T$, the subset $C_{\dot{w}}t$ is the symplectic leaf through the point $\dot{w}t$. As w runs over W and t over T, these are all the symplectic leaves in K;

d) Both left and right translations by elements in T leave π invariant;

e) The image of π under the projection map $K \to K/T$ is a well defined bi-vector field on K/T which we still denote by π. It defines a Poisson structure on K/T called the Bruhat-Poisson structure;

f) Symplectic leaves of the Bruhat-Poisson structure on K/T are precisely the Bruhat cells Σ_w, for $w \in W$, in K/T (and thus the name);

g) With respect to the left translations by elements in K, the Bruhat-Poisson structure is T-invariant but not K-invariant; The action map

$$K \times K/T \longrightarrow K/T: \ (k_1, \ k_2/T) \longmapsto k_1 k_2/T$$

is a Poisson map, making $(K/T, \ \pi)$ into a Poisson homogeneous $(K, \ \pi)$-space.

Therefore, for each $w \in W$ with a fixed representative $\dot{w}$ in K, both $C_{\dot{w}}$ and Σ_w inherit symplectic structures as symplectic leaves in K and K/T respectively, and the projection from $C_{\dot{w}}$ to Σ_w is a symplectic diffeomorphism. Moreover, the symplectic structure on Σ_w is invariant under the action of T by left translations. The goal of this section is to write down both the

symplectic structure - thus also the Liouville measure - on Σ_w and the moment map for the T-action in the $\{z_1,\ \bar{z}_1,\ z_2,\ \bar{z}_2,\ \ldots,z_l,\ \bar{z}_l\}$ coordinates. Here, we regard $\{z_1,\ \bar{z}_1,\ z_2,\ \bar{z}_2,\ \ldots,z_l,\ \bar{z}_l\}$ as coordinates on Σ_w via the parametrization of Σ_w by N_w,

Property b) of π on K is called the multiplicativity of π. As we will see shortly, it is exactly this property that enables us to decompose, as symplectic manifolds, the higher dimensional $C_{\dot{w}}$'s into products of 2-dimensional ones. In fact, this is also the motivation for introducing the coordinates $\{z_1,\ \bar{z}_1,\ z_2,\ \bar{z}_2,\ \ldots,z_l,\ \bar{z}_l\}$.

As in Section 2, let $l = l(w)$ and fix a reduced decomposition

$$w \ = \ \sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_l}.$$

For each $j = 1,...,l$, let $\dot{\gamma}_j$ be a representative of σ_{γ_j} in K, so $\dot{w} = \dot{\gamma}_1\dot{\gamma}_2\cdots\dot{\gamma}_l$ is a representative of w in K. Property c) of π on K says that the symplectic leaf through $\dot{\gamma}_j$ is the 2-dimensional cell $C_{\dot{\gamma}_j}$. Theorem 2.1 (see Remark 2.2) says that the map

$$C_{\dot{\gamma}_1}\times C_{\dot{\gamma}_1}\times\cdots\times C_{\dot{\gamma}_l}\longrightarrow C_{\dot{w}}. \tag{27}$$

is a diffeomorphism.

Proposition 3.2 *The map in (27) is a symplectic diffeomorphism, where the left hand side has the product symplectic structure.*

Proof. (Notice how the multiplicativity of π is used in the proof.) The inclusion map

$$C_{\dot{\gamma}_1}\times C_{\dot{\gamma}_1}\times\cdots\times C_{\dot{\gamma}_l}\ \hookrightarrow\ K\times K\times\cdots\times K \qquad (\ l\text{-copies})$$

is a Poisson map because each $C_{\dot{\gamma}_j}$ is a symplectic leaf and thus a Poisson submanifold of K. The multiplicativity of π says that the multiplication map

$$K\times K\times\cdots\times K\longrightarrow K:\ (k_1,\ k_2,\ \cdots,\ k_l)\longmapsto k_1k_2\cdots k_l$$

is a Poisson map. Thus, composing the two, we see that

$$C_{\dot{\gamma}_1}\times C_{\dot{\gamma}_1}\times\cdots\times C_{\dot{\gamma}_l}\longrightarrow K:\ (k_1,\ k_2,\ \cdots,\ k_l)\longmapsto k_1k_2\cdots k_l$$

is a Poisson map. But it has its image in $C_{\dot{w}}$ which is a Poisson submanifold of K. Thus, regarded as a map to $C_{\dot{w}}$, the above is a Poisson map, and thus a Poisson and therefore a symplectic diffeomorphism.

Q.E.D.

It now remains to determine the symplectic structure on the 2-dimensional leaves. Recall that for each simple root γ, we have a Lie group homomorphism

$$\Phi_\gamma : \ SL(2,\ \mathbb{C}) \longrightarrow G$$

which maps $SU(2)$ to K.

Proposition 3.3 *(See also [S]) For each simple root γ, equip $SU(2)$ with the Poisson structure given by*

$$\pi_\gamma \ = \ \Lambda_\gamma^r \ - \ \Lambda_\gamma^l,$$

where

$$\Lambda_\gamma \ = \ \frac{1}{4} \ll \gamma,\ \gamma \gg \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \ \in \ su(2) \wedge su(2),$$

and Λ_γ^r (resp. Λ_γ^l) denotes the right (resp. left) invariant bi-vector field on $SU(2)$ with value Λ_γ at the identity element. Then 1) $\Psi_\gamma :\ (SU(2),\ \pi_\gamma) \to (K,\ \pi)$ is a Poisson map, and 2) the symplectic leaf of π_γ in $SU(2)$ through the point $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ is

$$C_0 \ = \ \left\{ \begin{pmatrix} \frac{iz}{\sqrt{1+|z|^2}} & \frac{i}{\sqrt{1+|z|^2}} \\ \frac{i}{\sqrt{1+|z|^2}} & \frac{-i\bar{z}}{\sqrt{1+|z|^2}} \end{pmatrix} :\ z \in \mathbb{C} \right\}.$$

Using $(z,\ \bar{z})$ as coordinates on C_0, the induced symplectic structure is given by

$$\Omega \ = \ \frac{i}{\ll \gamma,\ \gamma \gg} \, \frac{1}{1+|z|^2} dz \wedge d\bar{z}.$$

Proof. 1) Think of $\mathfrak{a} + \mathfrak{n}$ as a real Lie algebra and identify $\mathfrak{a} + \mathfrak{n}$ with $\mathfrak{k}^*$ using the imaginary part of the Killing form as the pairing. The fact that Ψ_γ is a Poisson map then follows from the fact that the subspace $\Psi_\gamma(su(2))^\perp$ of $\mathfrak{a} + \mathfrak{n}$ consisting of all elements that annihilate $\Psi_\gamma(su(2))$ is an ideal of $\mathfrak{a} + \mathfrak{n}$. See [L-W].

2) As a special case of Theorem 3.1 (up to a constant multiple), the symplectic leaves of π_γ in $SU(2)$ are either 2 or 0-dimensional. We know from Section 2 that the set C_0 is the symplectic leaf through the point $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$.

Write an element of $SU(2)$ as

$$u \ = \ \begin{pmatrix} \xi & \eta \\ -\bar{\eta} & \bar{\xi} \end{pmatrix}.$$

The Poisson brackets defined by π_γ are

$$
\begin{aligned}
\{\xi,\ \bar{\xi}\} &= -i \ll \gamma,\ \gamma \gg |\eta|^2 & \{\xi,\ \eta\} &= \frac{1}{2} i \ll \gamma,\ \gamma \gg \xi\eta \\
\{\xi,\ \bar{\eta}\} &= \frac{1}{2} i \ll \gamma,\ \gamma \gg \xi\bar{\eta} & \{\eta,\ \bar{\eta}\} &= 0.
\end{aligned}
$$

Set

$$\xi = \frac{iz}{\sqrt{1+|z|^2}}, \qquad \eta = \frac{i}{\sqrt{1+|z|^2}},$$

so

$$z = \xi/\eta, \qquad \bar{z} = \bar{\xi}/\bar{\eta}.$$

Using the Leibniz rule for the Poisson bracket, we get

$$
\begin{aligned}
\{z,\ \bar{z}\} &= \{\xi/\eta,\ \bar{\xi}/\bar{\eta}\} \\
&= \frac{1}{|\eta|^2}\{\xi,\ \bar{\xi}\} - \frac{\bar{\xi}}{|\eta|^2\bar{\eta}}\{\xi,\ \bar{\eta}\} - \frac{\xi}{|\eta|^2\eta}\{\eta,\ \bar{\xi}\} \\
&= -i \ll \gamma,\ \gamma \gg (1 + \frac{|\xi|^2}{|\eta|^2}) \\
&= -i \ll \gamma,\ \gamma \gg (1+|z|^2).
\end{aligned}
$$

Thus the symplectic 2-form Ω on C_0 is given as stated.

Q.E.D.

Theorem 3.4 *In the $\{z_1,\ \bar{z}_1,\ z_2,\ \bar{z}_2,\ \ldots,z_l,\ \bar{z}_l\}$ coordinates, the induced symplectic structure on Σ_w (as a symplectic leaf of the Bruhat-Poisson structure) and the Liouville volume form μ_w associated to Ω_w are given by*

$$\Omega_w = \sum_{j=1}^{l} \frac{i}{\ll \alpha_j,\ \alpha_j \gg} \frac{1}{1+|z_j|^2}\, dz_j \wedge d\bar{z}_j \tag{28}$$

$$\mu_w := \frac{1}{l!}\,(\Omega_w)^l = \prod_{j=1}^{l} \frac{i}{\ll \alpha_j,\ \alpha_j \gg} \frac{1}{1+|z_j|^2}\, dz_j \wedge d\bar{z}_j. \tag{29}$$

The moment map for the T-action on Σ_w by left translations satisfying $\phi_w(0) = 0$ is given by

$$\phi_w :\ \Sigma_w \longrightarrow \mathfrak{t}^* \cong \mathfrak{a} :\ \phi_w = \sum_{j=1}^{l}(-\frac{1}{2}\log(1+|z_j|^2)\check{H}_{\alpha_j}). \tag{30}$$

Here we are identifying $\mathfrak{t}^*$ with $\mathfrak{a}$ by using the imaginary part of the Killing form as the pairing.

Proof. The formula for Ω_w follows immediately from Propositions 3.2 and 3.3 and the fact that $\ll \gamma_j,\ \gamma_j \gg = \ll \alpha_j,\ \alpha_j \gg$.

The formula for the moment ϕ_w follows from the explicit formula for the T-action on Σ_w in the $\{z_1,\ \bar{z}_1,\ z_2,\ \bar{z}_2,\ \ldots,z_l,\ \bar{z}_l\}$ coordinates as given in Proposition 2.11: let $iH \in \mathfrak{t}$, where $H \in \mathfrak{a}$. By Proposition 2.11, the generating vector field of the T action on Σ_w in the direction of iH is given by

$$V_{iH} = \sum_{j=1}^{l} \alpha_j(H)(-y_j \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial y_j}).$$

Set

$$V_{iH} \lrcorner \Omega_w = d < \phi_w,\ iH > .$$

We see that the only solution of ϕ_w that satisfies this equation for all $H \in \mathfrak{a}$ and the condition that $\phi_w(0) = 0$ is given by (30).

Q.E.D.

The following two corollaries follow immediately by comparing the formulas in Theorems 2.5, 2.7 and 3.4 (see also identity (20)).

Corollary 3.5 *Think of the moment map ϕ_w as a map from N_w to $\mathfrak{a}$ via the parametrization of Σ_w by N_w. Then*

$$\phi_w = Ad_{\dot{w}} \log a_w(n).$$

Corollary 3.6 *Think of the Haar measure dn of N_w as a volume form on Σ_w via the parametrization of Σ_w by N_w. It is related to the Liouville measure μ_w by*

$$\mu_w = \left(\prod_{j=1}^{l} \frac{\pi}{\ll \rho,\ \beta_j \gg} \right) a_w(n)^{-2\rho} dn.$$

We have thus connected the moment map ϕ_w and the Liouville measure μ_w with the familiar map $a_w : N_w \to A$ and the Haar measure dn on N_w. Such a connection is desirable for understanding the geometry of the Bruhat-Poisson structure. We have arrived at this by comparing their formulas in coordinates. This is why we wanted to write down the formulas for a_w and dn in the $\{z_1,\ \bar{z}_1,\ z_2,\ \bar{z}_2,\ \ldots,z_l,\ \bar{z}_l\}$ coordinates in Section 2.

4 Kostant's theorem on $H(G/B)$

In [K], Kostant constructs, for each element w in the Weyl group W, an explicit K-invariant closed differential form s^w on X with $\deg(s^w) = 2l(w)$, such that the cohomology classes of the s^w's form a basis of $H(X, \mathbb{C})$ that, up to scalar multiples, is dual to the basis of the homology of X formed by the closures of the Bruhat cells in X. We now recall the definition of these forms in more details.

Let $C = \oplus C^\bullet$ be the space of all K-invariant differential forms on $G/B \cong K/T$. By identifying $\mathfrak{g}$ and $\mathfrak{g}^*$ via the Killing form $\ll\ ,\ \gg$ so that

$$(\mathfrak{g}/\mathfrak{h})^* \ \cong\ \mathfrak{n}_- \ +\ \mathfrak{n},$$

we can identify

$$C^\bullet \ \cong\ \wedge^\bullet(\mathfrak{n}_- \ +\ \mathfrak{n})^T.$$

Introduce the operators E and L_0 on $C^\bullet$ by

$$E \ =\ 2\sum_{\alpha>0} ad_{E_{-\alpha}} \otimes ad_{E_\alpha}$$

and

$$\begin{aligned} &L_0(E_{-\alpha_1} \wedge E_{\alpha_2} \wedge \cdots \wedge E_{-\alpha_p} \otimes E_{\beta_1} \wedge E_{\beta_2} \wedge \cdots \wedge E_{\beta_q}) \\ =& \begin{cases} 0 & \text{if } \|\rho\|^2 - \|\rho - (\alpha_1 + \alpha_2 + \cdots + \alpha_p)\|^2 = 0 \\ \dfrac{1}{\|\rho\|^2 - \|\rho - (\alpha_1 + \alpha_2 + \cdots + \alpha_p)\|^2} & \text{if } \|\rho\|^2 - \|\rho - (\alpha_1 + \alpha_2 + \cdots + \alpha_p)\|^2 \neq 0 \end{cases} \end{aligned}$$

Set

$$R \ =\ -L_0 E.$$

It is a nilpotent operator.

Now let $w \in W$ be a Weyl group element. As before, use l to denote $l(w)$. Let

$$\{\alpha_1,\ \alpha_2,\ ...,\ \alpha_l\} \ =\ \{\alpha > 0 :\ w^{-1}\alpha < 0\},$$

and let

$$\beta_j \ =\ -w^{-1}\alpha_j$$

so that

$$\{\beta_1,\ \beta_2, ...,\ \beta_l\} \ =\ \{\beta > 0 :\ w\beta < 0\}.$$

Set

$$h^{w^{-1}} \;=\; (\frac{i}{2})^l \; E_{-\beta_1} \wedge E_{-\beta_2} \wedge \cdots \wedge E_{-\beta_l} \otimes E_{\beta_1} \wedge E_{\beta_2} \wedge \cdots \wedge E_{\beta_l},$$

and

$$s^w \;=\; (1-R)^{-1} h^{w^{-1}} \;=\; h^{w^{-1}} \;+\; R h^{w^{-1}} \;+\; R^2 h^{w^{-1}} \;+\; \cdots . \tag{31}$$

Theorem 4.1 (Kostant [K]) *1). The forms s^w for $w \in W$ are closed, and their cohomology classes form a basis of the de Rham cohomology of $G/B \cong K/T$ (with complex coefficients);*

2). Let $j_w: \; N_w \to \Sigma_w : n \mapsto nw/T$ be the parametrization map. Then

$$j_{w_1}^*(s^w|_{\Sigma_{w_1}}) \;=\; \begin{cases} 0 & \text{if } l(w_1) = l(w) \text{ but } w_1 \neq w \\ a_w(n)^{-2(\rho - w^{-1}\rho)} (dn)_1 & \text{if } w_1 = w, \end{cases}$$

where $s^w|_{\Sigma_{w_1}} = i_{w_1}^* s^w$ if i_{w_1} is the inclusion map of Σ_{w_1} into K/T, the map $a_w: N_w \to A$ is, as before, defined by (4), and $(dn)_1$ is the left invariant Haar measure on N_w normalized by the condition

$$((dn)_1, \; E_{\alpha_1} \wedge iE_{\alpha_1} \wedge E_{\alpha_2} \wedge iE_{\alpha_2} \wedge \cdots \wedge E_{\alpha_l} \wedge iE_{\alpha_l}) \;=\; 1.$$

3)

$$\int_{\Sigma_w} s^w \;=\; \prod_{j=1}^{l} \frac{\pi}{\ll \rho, \; \alpha_j \gg}.$$

The purpose of this section is to relate the form s^w with the Liouville volume form μ_w on Σ_w induced by the Bruhat-Poisson structure. This is now easy due to Corollaries 3.5 and 3.6. We will also give a simple proof of 3).

We first need to relate the left Haar measures $(dn)_1$ and dn as given in Theorem 2.7.

Lemma 4.2 *We have*

$$(dn)_1 \;=\; \prod_{j=1}^{l} \frac{\pi}{\ll \rho, \; \beta_j \gg} dn.$$

Proof. This follows from Proposition 2.10 and the fact that $\ll \alpha_j, \; \alpha_j \gg = \ll \beta_j, \; \beta_j \gg$ for each $j = 1, ..., l$.

Q.E.D.

Theorem 4.3 *When restricted to the Schubert cell Σ_w, Kostant's harmonic form s^w is related to the Liouville volume form μ_w on Σ by*

$$s^w|_{\Sigma_w} \;=\; (a_w)^{2w^{-1}\rho}\mu_w \;=\; e^{<\phi_w,\; 2iH_\rho>}\mu_w. \tag{32}$$

Proof. This is a direct consequence of Corollaries 3.5 and 3.6. Explicitly, we have

$$s^w|_{\Sigma_w} \;=\; \prod_{j=1}^{l} \frac{i}{\ll \alpha_j,\; \alpha_j \gg}(1+|z_j|^2)^{-\dfrac{2\ll \rho,\; \alpha_j \gg}{\ll \alpha_j,\; \alpha_j \gg}\;-\;1} dz_j \wedge d\bar{z}_j \tag{33}$$

$$< \phi_w,\; 2iH_\rho > \;=\; -\sum_{j=1}^{l} \frac{2\ll \rho,\; \alpha_j \gg}{\ll \alpha_j,\; \alpha_j \gg}\log(1+|z_j|^2) \tag{34}$$

and

$$\mu_w \;=\; \prod_{j=1}^{l} \frac{i}{\ll \alpha_j,\; \alpha_j \gg}(1+|z_j|^2)^{-1} dz_j \wedge d\bar{z}_j.$$

Thus we have (32).

Q.E.D.

From the explicit formula for $s^w|_{\Sigma_w}$, we immediately get

$$\begin{aligned}
\int_{\Sigma_w} s^w \;&=\; \prod_{j=1}^{l} \int_{\mathbb{R}^2} \frac{i}{\ll \alpha_j,\; \alpha_j \gg}(1+|z_j|^2)^{-\dfrac{2\ll \rho,\; \alpha_j \gg}{\ll \alpha_j,\; \alpha_j \gg}\;-\;1} dz_j \wedge d\bar{z}_j \\
&=\; \prod_{j=1}^{l} \frac{\pi}{\ll \rho,\; \alpha_j \gg}.
\end{aligned}$$

This integral was first calculated in [K-K] using induction on $l(w)$. Again, as in the case of the c-functions, our simple proof is due to the fact that the induction argument has been pushed to the calculations for $a_w(n)$ and dn. This is in fact a special case of the c-function with $i\lambda = -2w^{-1}\rho$.

Remark 4.4 When $w = w_0$ is the longest Weyl group element, the form s^{w_0} is a K-invariant volume form so it coincides with the Haar measure on K/T. When restricted to the biggest cell Σ_{w_0}, the Liouville volume form μ_{w_0}, the form s^{w_0}, and the Haar measure dn for $N_{w_0} = N$ are related by

$$s^{w_0}|_{\Sigma_{w_0}} \;=\; (a_{w_0})^{-2\rho}\mu_{w_0} \;=\; \left(\prod_{\alpha>0} \frac{\pi}{\ll \rho,\; \alpha \gg}\right) a_{w_0}(n)^{-4\rho} dn.$$

Remark 4.5 The function $< \phi_w,\ 2iH_\rho >$ on Σ_w is the Hamiltonian function for the generating vector field θ_0 of the T-action in the direction of $2iH_\rho$. This vector field is intrinsic to the Bruhat-Poisson structure in the sense that it is its modular vector field. As $|z_j| \to +\infty$, the function $< \phi_w,\ 2iH_\rho > \to -\infty$. Thus the modular vector field θ_0 is not globally Hamiltonian on K/T. We say that the Bruhat-Poisson structure is not unimodular. See [W2] [B-Z] [E-L-W].

Remark 4.6 The second identity in (32) expresses the form $s^w|_{\Sigma_w}$ totally in terms of data coming from the Bruhat-Poisson structure. In particular, it says that the integral $\int_{\Sigma_w} s^w$ is of the Duistermaat-Heckman type. Wanting to see this was another motivation for this work. Theorem 4.3 can be used to describe generators of the so-called Poisson - de Rham cohomology of the Bruhat-Poisson structure. We do this in [E-L].

5 Appendix: Relation to the Bott-Samelson coordinates

As before, let $w \in W$ be a Weyl group element with a fixed reduced decomposition:

$$w \ = \ \sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_l}$$

where $l = l(w)$. Then we have the Lie group homomorphism $\Psi_{\gamma_j} : SL(2,\mathbb{C}) \to G$ and the element

$$\dot{\gamma}_j \ = \ \Psi_{\gamma_j}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \ \in \ K$$

for each $j = 1, ..., l$. Set again $\dot{w} = \dot{\gamma}_1\dot{\gamma}_2\cdots\dot{\gamma}_l \in K$. Then the map

$$\begin{aligned} F'_w :\ & N_{\gamma_1} \times \cdots \times N_{\gamma_l} \longrightarrow N_w : \\ (n_1,\ n_2,\ \dots ,n_l) & \longmapsto n_1\dot{\gamma}_1 n_2\dot{\gamma}_2 \cdots n_l\dot{\gamma}_l\dot{w}^{-1} \\ & = \ n_1\ (\dot{\gamma}_1 n_2 \dot{\gamma}_1^{-1}) \cdots (\dot{\gamma}_1\dot{\gamma}_2\cdots\dot{\gamma}_{l-1})\ n_l\ (\dot{\gamma}_1\dot{\gamma}_2\cdots\dot{\gamma}_{l-1})^{-1} \end{aligned}$$

is a holomorphic diffeomorphism. Thus, if we parametrize N_{γ_j} by $\mathbb{C}$ via

$$z'_j \longmapsto n(z'_j) \ := \ \Psi_{\gamma_j}\begin{pmatrix} 1 & z'_j \\ 0 & 1 \end{pmatrix} \ = \ \exp(z'_j\check{E}_{\gamma_j}) \ \in \ N_{\gamma_j}$$

for each j, we get a parametrization of N_w by $\mathbb{C}^l$:

$$\mathbb{C}^l \longrightarrow N_w :\ (z'_1,\ z'_2,\ \dots ,z'_l) \longmapsto F'_w(n(z'_1),\ n(z'_2),\ \dots ,n(z'_l)).$$

We call it the Bott-Samelson parametrization of N_w and call $\{z'_1,\ z'_2,\ \dots ,z'_l\}$ the **Bott-Samelson coordinates** on N_w because of the close relation to the Bott-Samelson desingularization of the Schubert variety X_w [J].

The difference between our coordinates $\{z_1,\ \bar{z}_1,\ z_2,\ \bar{z}_2,\ \dots,z_l,\ \bar{z}_l\}$ and the Bott-Samelson coordinates $\{z_1',\ z_2',\ \dots,z_l'\}$ can be described as follows: by composing F_w (in Theorem 2.1) and F_w' with the map

$$j_w:\ N_w \longrightarrow \Sigma_w:\ n \longmapsto n\dot{w}/B,$$

we can think of both F_w and F_w' as mapping $N_{\gamma_1} \times \cdots \times N_{\gamma_l}$ diffeomorphically to Σ_w. The map F_w' first sends $(n_1, n_2, \dots, n_l)$ to the product of $n_1\dot{\gamma}_1 n_2\dot{\gamma}_2 \cdots n_l\dot{\gamma}_l$ in G and then projects the product to G/B. But for F_w, we first pick up the K-component k_j in the Iwasawa decomposition of $n_j\dot{\gamma}_j$ for each j, multiply the k_j's inside K and then project the product to $K/T \cong G/B$.

Consider now the change of coordinates

$$(z_1',\ z_2',\ \dots,\ z_l') \ = \ I_w(z_1,\ z_2, \dots,\ z_l).$$

We can get a recursive formula for I_w from the proof of Theorem 2.7. Clearly, when w is a simple reflection, the map I_w is the identity map. For a general w with the reduced decomposition $w = \sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_l}$, let again $w_1 = \sigma_{\gamma_1}\sigma_{\gamma_2}\cdots\sigma_{\gamma_{l-1}}$ so that $w = w_1\sigma_{\gamma_l}$. Keeping the same notation as in the proof of Theorem 2.7, we know that I_w is given by

$$\begin{cases} (z_1',\ z_2',\ \dots,\ z_{l-1}') \ = \ I_{w_1}(z_1,\ z_2, \dots,\ z_{l-1}) \\ z_l' \ = \ m(z_1,\ z_2, \dots,\ z_{l-1}) \ + \ a_{w_1}(n)^{-\gamma_l} z_l, \end{cases}$$

where, recall, $m(z_1,\ z_2, \dots,\ z_{l-1}) \in \mathbb{C}$ is such that $\exp(m(z_1, z_2, \dots, z_{l-1})\check{E}_{\gamma_l})$ is the N_{γ_l}-component of $(m')^{-1} \in N$ with respect to the decomposition $N = N_{\gamma_l}N_{\hat{\gamma}_l}$, and $m' \in N$ is the N-component in the Iwasawa decomposition of $n\dot{w}_1$. The change of coordinates is in general not complex because the function $m(z_1,\ z_2, \dots,\ z_{l-1})$ is in general not holomorphic. This is also seen from the following example.

Example 5.1 For $G = SL(3, \mathbb{C})$ and $w = (1,2)(2,3)(1,2)$, the Bott-Samelson parametrization of N_w is by

$$(z_1',\ z_2',\ z_3') \longmapsto \begin{pmatrix} 1 & z_1' & z_1'z_3' + iz_2' \\ 0 & 1 & z_3' \\ 0 & 0 & 1 \end{pmatrix}.$$

The change of coordinates between $(z_1,\ z_2,\ z_3)$ and $\{z_1',\ z_2',\ z_3'\}$ are (see Example 2.9)

$$z_1' \ = \ z_1, \qquad z_2' \ = \ \varepsilon_1 z_2 \qquad z_3' \ = \ \frac{\varepsilon_2 z_3 - i\bar{z}_1 z_2}{\varepsilon_1},$$

or

$$z_1 \ = \ z_1', \qquad z_2 \ = \ \frac{z_2'}{\eta_1} \qquad z_3 \ = \ \frac{\eta_1^2 z_3' + i\overline{z_1'}z_2'}{\eta_2},$$

where $\varepsilon_j = \sqrt{1 + |z_j|^2}$ for $j = 1, 2$ and $\eta_1 = \sqrt{1 + |z_1'|^2},\ \eta_2 = \sqrt{1 + |z_1'|^2 + |z_2'|^2}$.

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